

# general scheme for super dense coding between multi-parties

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(Dated: today)

Dense coding or super-dense coding in the case of high-dimension quantum states between two parties and multi-parties has been studied in this paper. We construct explicitly the measurement basis and the forms of the single-body unitary operations corresponding to the basis chosen, and the rules for selecting the one-body unitary operations in a multi-party case.

PACS numbers: 03.67.-a, 89.70.+c

Published in Phys. Rev. A65 (2002) 022304

Quantum dense-coding or super-dense coding[1] is one of the important branches of quantum information theory. It has been widely studied both in theory and in experiment[1, 2]. The basic idea of quantum dense coding is that quantum mechanics allows one to encode information in the quantum states that is more dense than classical coding. Bell-basis states

$$\begin{aligned} |\Psi^+\rangle &= (|00\rangle + |11\rangle)/\sqrt{2}, \\ |\Psi^-\rangle &= (|00\rangle - |11\rangle)/\sqrt{2}, \\ |\phi^+\rangle &= (|01\rangle + |10\rangle)/\sqrt{2}, \\ |\phi^-\rangle &= (|01\rangle - |10\rangle)/\sqrt{2}. \end{aligned} \quad (1)$$

are used in dense coding. Bell basis states are in the Hilbert space of 2 particles, each with 2-dimension, and they are the maximally entangled states. Supposed Alice and Bob share the maximally entangled state  $|\Psi^+\rangle$ . Bob then operates locally on the particle he shares with Alice one of the four unitary transformations  $I, \sigma_x, i\sigma_y, \sigma_z$ , and this will transform  $|\Psi^+\rangle$  into  $|\Psi^+\rangle, |\phi^+\rangle, |\phi^-\rangle$  and  $|\Psi^-\rangle$  respectively. Bob sends his particle back to Alice. Because the four manipulations result in four orthogonal Bell states, four distinguishable messages, i.e., 2 bits of information then can be obtained by Alice via collective-measurement on the two particles. The scheme has been experimentally demonstrated by Mattle et. al.[3].

With the realization of preparing high-dimension quantum state [4], it is of practical importance to study the high-dimensional aspects of various topics in quantum information. For example, multi-particle high-dimensional quantum teleportation has been constructed recently[5]. Teleportation and quantum dense coding are closely related. In this paper, we presents a quantum dense coding scheme between multi-parties in an arbitrary high dimensional Hilbert space. As two-party dense coding is of primary importance, we first present the two-party dense coding scheme in arbitrary high dimension. Then we present the general scheme for dense coding between multi-parties using high dimensional state.

To present our scheme clearer. Let's first begin with dense coding between two parties in 3-dimension. The general Bell-basis of the Hilbert space of two particles with 3-dimension are [5, 6]:

$$|\Psi_{nm}\rangle = \sum_j e^{2\pi i j n/3} |j\rangle \otimes |j + m \bmod 3\rangle / \sqrt{3}, \quad (2)$$

where  $n, m, j = 0, 1, 2$ . Explicitly,

$$\begin{aligned} |\Psi_{00}\rangle &= (|00\rangle + |11\rangle + |22\rangle)/\sqrt{3}, \\ |\Psi_{10}\rangle &= (|00\rangle + e^{2\pi i/3}|11\rangle + e^{4\pi i/3}|22\rangle)/\sqrt{3}, \\ |\Psi_{20}\rangle &= (|00\rangle + e^{4\pi i/3}|11\rangle + e^{2\pi i/3}|22\rangle)/\sqrt{3}, \\ |\Psi_{01}\rangle &= (|01\rangle + |12\rangle + |20\rangle)/\sqrt{3}, \\ |\Psi_{11}\rangle &= (|01\rangle + e^{2\pi i/3}|12\rangle + e^{4\pi i/3}|20\rangle)/\sqrt{3}, \\ |\Psi_{21}\rangle &= (|01\rangle + e^{4\pi i/3}|12\rangle + e^{2\pi i/3}|20\rangle)/\sqrt{3}, \\ |\Psi_{02}\rangle &= (|02\rangle + |10\rangle + |21\rangle)/\sqrt{3}, \\ |\Psi_{12}\rangle &= (|02\rangle + e^{2\pi i/3}|10\rangle + e^{4\pi i/3}|21\rangle)/\sqrt{3}, \\ |\Psi_{22}\rangle &= (|02\rangle + e^{4\pi i/3}|10\rangle + e^{2\pi i/3}|21\rangle)/\sqrt{3}. \end{aligned} \quad (3)$$

Superdense coding can be done in the following way. Supposed Alice and Bob share the maximally entangled state  $|\Psi_{00}\rangle$ . Through simple calculation, it can be shown that the single-body operators :

$$\begin{aligned}
U_{00} &= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \\
U_{10} &= \begin{bmatrix} 1 & 0 & 0 \\ 0 & e^{2\pi i/3} & 0 \\ 0 & 0 & e^{4\pi i/3} \end{bmatrix}, \\
U_{20} &= \begin{bmatrix} 1 & 0 & 0 \\ 0 & e^{4\pi i/3} & 0 \\ 0 & 0 & e^{2\pi i/3} \end{bmatrix}, \\
U_{01} &= \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}, \\
U_{11} &= \begin{bmatrix} 0 & 0 & e^{4\pi i/3} \\ 1 & 0 & 0 \\ 0 & e^{2\pi i/3} & 0 \end{bmatrix}, \\
U_{21} &= \begin{bmatrix} 0 & 0 & e^{2\pi i/3} \\ 1 & 0 & 0 \\ 0 & e^{4\pi i/3} & 0 \end{bmatrix}, \\
U_{02} &= \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}, \\
U_{12} &= \begin{bmatrix} 0 & e^{2\pi i/3} & 0 \\ 0 & 0 & e^{4\pi i/3} \\ 1 & 0 & 0 \end{bmatrix}, \\
U_{22} &= \begin{bmatrix} 0 & e^{4\pi i/3} & 0 \\ 0 & 0 & e^{2\pi i/3} \\ 1 & 0 & 0 \end{bmatrix}, \tag{4}
\end{aligned}$$

will transform  $|\Psi_{00}\rangle$  into the corresponding states in Eq. (3) respectively:

$$U_{nm}|\Psi_{00}\rangle = |\Psi_{nm}\rangle. \tag{5}$$

Bob operates one of the above unitary transformations, and sends his particle back to Alice. Alice takes only one measurement in the basis  $\{|\Psi_{00}\rangle, |\Psi_{10}\rangle, \dots, |\Psi_{22}\rangle\}$ , and she will know what operation Bob has done, or to say, what the messages Bob has encoded in the quantum state. As a result, Alice gets  $\log_2 9$  bits information through only one measurement. It should be pointed out that Michael Reck et. al.[8] have given the method to realize any discrete unitary operators and the operators used here can be constructed according to their protocol.

It is straightforward to generalize the above protocol to arbitrarily high dimension for two parties. We denote the dimension as  $d$ . The general Bell-basis states are

$$|\Psi_{nm}\rangle = \sum_j e^{2\pi i j n/d} |j\rangle \otimes |j + m \bmod d\rangle / \sqrt{d}, \tag{6}$$

where  $n, m, j = 0, 1, \dots, d-1$ . Obviously, there exist one-body operators  $U_{nm}$  on Bob's particle that satisfy  $U_{nm}|\Psi_{00}\rangle = |\Psi_{nm}\rangle$ . The matrix elements of the unitary transformation  $U_{nm}$  may be explicitly written out:

$$(U_{nm})_{j'j} = e^{2\pi i j n/d} \delta_{j', j+m \bmod d}, \tag{7}$$

where  $n, m, j = 0, 1, \dots, d-1$ . The procedure for realizing dense coding in this high dimension case is similar to that for the 2 dimensional case: Alice and Bob shares the state  $|\Psi_{00}\rangle$ , then Bob performs one of the operations in (7) to his particle and sends this particle back to Alice. Alice then performs a collective measurement in the basis states in (6) to find out what Bob has done to the particle and hence reads out the encoded message. In this case, Alice gets  $\log_2 d^2$  bits of information just making only one measurement.

Dense coding between two parties can be generalized into **multi-parties**. Bose et. al.[7] has generalized Bennett and Wiesner scheme of dense coding into multi-parties in the qubit system. The multi-party dense coding scheme can be understood in the following way. There are  $N+1$  users sharing an  $(N+1)$ -particle maximally entangled

state, possessing one particle each. Suppose that one of them, say user 1, intends to receive messages from the  $N$  other users. The  $N$  senders mutually decide *a priori* to perform only certain unitary operations on their particles. After performing their unitary operations, each of the  $N$  senders sends his particle to user 1. User 1 then performs a collective measurement on the  $N + 1$  particles and identifies the state. Thus he can learn about the operation of each of the other  $N$  users has done. That is to say that a single measurement is sufficient to reveal the messages sent by all the  $N$  users. We now discuss the high-dimension generalization of this scenario. We also begin with an example with three particles in 3-dimension. We take the maximally entangle states as our basis:

$$|\Psi_{nm}^k\rangle = \sum_j e^{2\pi i j k/3} |j\rangle \otimes |j + n \bmod 3\rangle \otimes |j + m \bmod 3\rangle / \sqrt{3}, \quad (8)$$

where  $n, m, k = 0, 1, 2$ . More explicitly:

$$\begin{aligned} |\Psi_{00}^0\rangle &= (|000\rangle + |111\rangle + |222\rangle) / \sqrt{3}, \\ |\Psi_{01}^0\rangle &= (|001\rangle + |112\rangle + |220\rangle) / \sqrt{3}, \\ |\Psi_{02}^0\rangle &= (|002\rangle + |110\rangle + |221\rangle) / \sqrt{3}, \\ &\dots, \\ |\Psi_{22}^2\rangle &= (|022\rangle + e^{4\pi i/3}|100\rangle + e^{2\pi i/3}|211\rangle) / \sqrt{3}. \end{aligned} \quad (9)$$

Suppose Alice, Bob and Claire share the maximally entangled state

$$|\Psi_{00}^0\rangle = (|000\rangle + |111\rangle + |222\rangle) / \sqrt{3}, \quad (10)$$

and Bob and Claire hold particle 2 and 3 respectively. The essential issue in dense coding is to find a limited number of one-body operations that Bob and Claire can perform so that the state  $|\Psi_{00}\rangle$  is transformed into all possible states in (9). Meanwhile these operations have to be able identified by Alice uniquely. If Bob and Claire are both allowed to perform any of the nine operations in (4), the total number of operations are  $9 \times 9 = 81$ , which is greater than the total number of the basis states in (9). Alice can not identify the operations of Bob and Claire uniquely. Thus not all the operations are allowed, and some restriction has to be made. In the qubit system, this problem is solved by allowing Bob to perform all 4 possible unitary operations  $I, \sigma_x, i\sigma_y$  and  $\sigma_z$ , and Claire only performs any two of these operations. However, direct generalization of this rule needs care. By direct calculation, we find that by allowing Bob to perform all the nine operations and restricting Claire to perform any 3 of the nine operations will not always work. Look at table I where we have given the results of the operations  $U_{nm}(B)U_{n'm'}(C)$  on  $|\Psi_{00}\rangle$ . Now allow Bob to perform all the nine unitary operations. Then from table I we see that the product of the nine operations of Bob with each operation in the subset  $\{U_{00}, U_{10}, U_{20}\}$  of Claire can only give 9 of the 27 basis states in (9). This is evident from table I that the first 3 columns in table (I) are just a rearrangement of the same 9 basis states. The same is true for the other two subsets of operations  $\{U_{01}, U_{11}, U_{21}\}$  and  $\{U_{02}, U_{12}, U_{22}\}$ . Thus, Claire's 3 operations can be chosen one from each of the 3 subsets arbitrarily. One such set is the three operations  $U_{nm}$  with  $n = 0$ :  $U_{00}, U_{01}$  and  $U_{02}$ . Other combinations are also possible. For convenience, we can simply choose this subset:  $\{U_{nm}$  with  $n = 0\}$  as the allowed operations of Claire.

With the identification of the allowed operations, say, Bob is allowed to perform all the nine unitary transformations, and Claire is allowed to perform the 3 operations  $U_{0m}$ ,  $m = 0, 1$  and  $2$ , the 3-party dense coding can be done easily. Bob and Claire perform their operations on their respective particles and return the particles to Alice. When Alice receives the operated particles from Bob and Claire, she can find what messages Bob and Claire have encoded by just one measurement in the basis of Eq. (9). That's, Alice gets  $\log_2 27$  bits information with a single measurement.

From the above example, we can generalize the scheme into multi-party super-dense coding in high dimensions. Suppose the dimension is  $d$ . We construct the following basis,

$$|\Psi_{i_1, i_2, \dots, i_N}^n\rangle = \sum_j e^{2\pi i j n/d} |j\rangle \otimes |j + i_1 \bmod d\rangle \otimes \dots \otimes |j + i_N \bmod d\rangle / \sqrt{d} \quad (11)$$

where  $n, j, i_1, i_2, \dots, i_N = 0, 1, \dots, d - 1$ . The  $d^2$  one-body unitary operations can be written explicitly

$$(U_{nm})_{j'j} = e^{2\pi i j n/dN} \delta_{j', j+m \bmod d}, \quad (12)$$

where  $j = 0, 1, \dots, d - 1$ . Thus, if the  $N$  senders mutually decide *a priori* that User 2 can perform all the  $d^2$  one-body operations in (12), User 3 to User  $N + 1$  can only perform the  $d$  one-body operations,  $U_{0m}$ ,  $m = 0, 1, \dots, d - 1$ , operations. Then the  $N$  users can encode their messages by performing their allowed operations to the particles at their proposal and return the particles to User 1. After User 1 receives all the particles, he can then perform a single collective measurement of all his  $N + 1$  particles so that he can reads out the messages the  $N$  users have encoded. In this way, the  $N$  users can send out  $\log_2 d^{N+1}$  bits messages and the receiver, user 1, gets and identifies them by only

TABLE I: Transformation table for  $U_{nm}(B)U_{n'm'}(C)\Psi_{00}$ 

$U_{nm}(B)$	$U_{n'm'}(C)$								
	$U_{00}$	$U_{10}$	$U_{20}$	$U_{01}$	$U_{11}$	$U_{21}$	$U_{02}$	$U_{12}$	$U_{22}$
$U_{00}$	$\Psi_{00}^0$	$\Psi_{00}^1$	$\Psi_{00}^2$	$\Psi_{01}^0$	$\Psi_{01}^1$	$\Psi_{01}^2$	$\Psi_{02}^0$	$\Psi_{02}^1$	$\Psi_{02}^2$
$U_{10}$	$\Psi_{00}^1$	$\Psi_{00}^2$	$\Psi_{00}^0$	$\Psi_{01}^1$	$\Psi_{01}^2$	$\Psi_{01}^0$	$\Psi_{02}^1$	$\Psi_{02}^2$	$\Psi_{02}^0$
$U_{20}$	$\Psi_{00}^2$	$\Psi_{00}^0$	$\Psi_{00}^1$	$\Psi_{01}^2$	$\Psi_{01}^0$	$\Psi_{01}^1$	$\Psi_{02}^2$	$\Psi_{02}^0$	$\Psi_{02}^1$
$U_{01}$	$\Psi_{10}^0$	$\Psi_{10}^1$	$\Psi_{10}^2$	$\Psi_{11}^0$	$\Psi_{11}^1$	$\Psi_{11}^2$	$\Psi_{12}^0$	$\Psi_{12}^1$	$\Psi_{12}^2$
$U_{11}$	$\Psi_{10}^1$	$\Psi_{10}^2$	$\Psi_{10}^0$	$\Psi_{11}^1$	$\Psi_{11}^2$	$\Psi_{11}^0$	$\Psi_{12}^1$	$\Psi_{12}^2$	$\Psi_{12}^0$
$U_{21}$	$\Psi_{10}^2$	$\Psi_{10}^0$	$\Psi_{10}^1$	$\Psi_{11}^2$	$\Psi_{11}^0$	$\Psi_{11}^1$	$\Psi_{12}^2$	$\Psi_{12}^0$	$\Psi_{12}^1$
$U_{20}$	$\Psi_{20}^0$	$\Psi_{20}^1$	$\Psi_{20}^2$	$\Psi_{21}^0$	$\Psi_{21}^1$	$\Psi_{21}^2$	$\Psi_{22}^0$	$\Psi_{22}^1$	$\Psi_{22}^2$
$U_{12}$	$\Psi_{20}^1$	$\Psi_{20}^2$	$\Psi_{20}^0$	$\Psi_{21}^1$	$\Psi_{21}^2$	$\Psi_{21}^0$	$\Psi_{22}^1$	$\Psi_{22}^2$	$\Psi_{22}^0$
$U_{22}$	$\Psi_{20}^2$	$\Psi_{20}^0$	$\Psi_{20}^1$	$\Psi_{21}^2$	$\Psi_{21}^0$	$\Psi_{21}^1$	$\Psi_{22}^2$	$\Psi_{22}^0$	$\Psi_{22}^1$

one measurement. This is because the  $N$  senders are allowed perform  $d^2 \times d \times \dots \times d = d^{N+1}$  different combinations of the one-body unitary operations on the initial  $(N+1)$ -particle maximally entangled state

$$|\Psi_{00\dots 0}\rangle = (|00\dots 0\rangle + |11\dots 1\rangle + \dots + |d-1, d-1\dots d-1\rangle)/\sqrt{d}, \quad (13)$$

while there are  $d^{N+1}$  orthogonal states in the Hilbert space of  $N+1$  particles.

In summary, we have given general schemes for multi-party high dimensional dense coding. Explicit expressions for the measuring basis and the forms of the one-body unitary transformation operators have been constructed. In particular, the operations allowed to users 3 to  $N$  must chosen carefully.

This work is supported by the Major State Basic Research Developed Program Grant No. G200077400, China National Natural Science Foundation Grant No. 60073009, the Fok Ying Tung Education Foundation, and the Excellent Young University Teachers' Fund of Education Ministry of China.

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